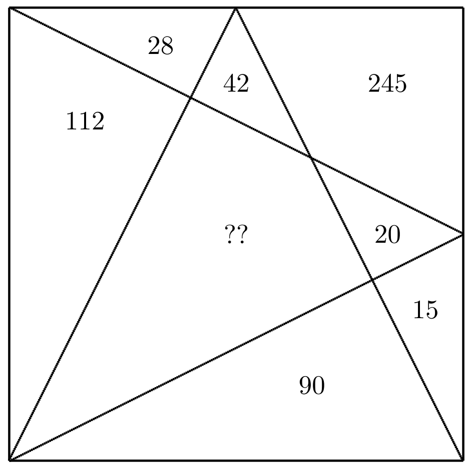
11^{th} Annual Johns Hopkins Math Tournament
Sunday, April 11, 2010
Grab Bag-Upper Division

(1) (**7**) Below is a square, divided by several lines (not to scale). Several regions have their areas written inside. Find the area of the remaining region.



(2) (**8**) A line is drawn tangent to the graph of $f(x) = \frac{1}{x}$ at the point $(a, f(a))$ in the first quadrant. The tangent line, x- and y-axes form a triangle. Find the area of the triangle in terms of a.

(3) (**10**) Let $x > 0$. If $\int_0^\infty e^{-t^2}\,dt = \frac{\sqrt{\pi}}{2}$, find $\Gamma\left(\frac{1}{2}\right)$ where $\Gamma(x)$ is the function defined by

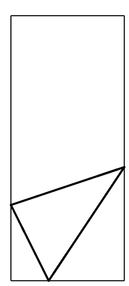
$$\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}\,dt$$

(4) (**12**) Fifteen chairs are lined up in a row for Professor Zucker's Honors Linear Algebra Exam. However, only 6 students show up and Zucker won't let any two students sit next to each other. In how many ways can Zucker arrange his students?

(5) (**13**) Evaluate the following limit:

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n+k}$$

(6) (**15**) The lower right-hand corner of a page is folded over so that it just touches the left edge of the paper, as shown in the figure below. If the width of the paper is α and the page is very long, find the minimum length of the crease if the lower left corner is held fixed.



(7) (**17**) Let $f(x) = x^6 - 3x^2 + x$. The graph of f has three real critical points. Find the unique quadratic equation which passes through these three points.

(8) (**18**) Evaluate the integral $\displaystyle\int_0^1 (e-1)\sqrt{\ln(1+ex-x)} + e^{x^2}\,dx$.